A NOMOGRAM FOR THE DETERMINATION OF SOLAR ALTITUDE AND AZIMUTH[1]

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ABSTRACT

A combination nomogram is presented for the calculation of solar altitude and azimuth for any latitude, solar declination, and time of day.

1. INTRODUCTION

Meteorologists, and other scientists or engineers working in allied fields, frequently wish to determine the solar altitude and/or azimuth for various purposes. The present paper describes the construction of a combination nomogram for the calculation of both these quantities for any latitude, solar declination, and time of day.

2. DERIVATION AND CONSTRUCTION OF THE SOLAR ALTITUDE NOMOGRAM

The altitude of the sun is given by

$$\sin\alpha=\sin\phi\sin\delta+\cos\phi\cos\delta\cos h \tag{1}$$

where:

α=solar altitude (angular elevation above the horizon)

ϕ=geographic latitude of the observer

δ=declination of the sun

h=hour angle of the sun

Equation (1) may be compared with the relationship among the coordinates of three points $P_1(X_1,Y_1)$, $P_3(X_3,Y_3)$ and $P_5(0,Y_5)$, lying in a straight line, as illustrated in figure 1.

$$Y_5\left\{\frac{X_1}{X_3}-1\right\}-Y_3\frac{X_1}{X_3}+Y_1=0. \tag{2}$$

If we set

$$Y_1=K_1\sin\alpha+C_1, \tag{3}$$

$$Y_5=K_5\cos h+C_5, \tag{4}$$

and substitute these in (1) for $\sin\alpha$ and $\cos h$, we get

$$Y_5\left\{-\frac{K_1}{K_5}\cos\phi\cos\delta\right\}-\left\{C_1+K_1\sin\phi\sin\delta-C_5\frac{K_1}{K_5}\cos\phi\cos\delta\right\}+Y_1=0. \tag{5}$$

If we now postulate that the left hand sides of (2) and (5) are to be identical, then

$$\frac{X_1}{X_3}-1=-\frac{K_1}{K_5}\cos\phi\cos\delta \tag{6}$$

$$Y_3\frac{X_1}{X_3}=C_1+K_1\sin\phi\sin\delta-\frac{C_5K_1}{K_5}\cos\phi\cos\delta. \tag{7}$$

In summary, the following equations define the nomogram

$$Y_1=K_1\sin\alpha+C_1$$

$$Y_5=K_5\cos h+C_5$$

$$\frac{X_1}{X_3}=1-\frac{K_1}{K_5}\cos\phi\cos\delta \tag{8}$$

$$Y_3=C_5+\frac{X_3}{X_1}(C_1-C_5+K_1\sin\phi\sin\delta).$$

K_1, C_1, K_5, C_5, X_1 are arbitrary factors, each having the dimension of length, which determine the size of the nomogram and the relative positions of the various scales.

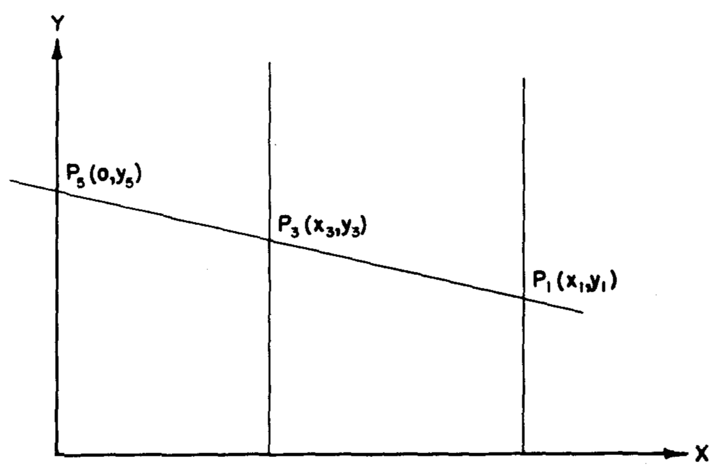

FIGURE 1.—Relationship among coordinates of three points lying in a straight line, similar to equation (2).

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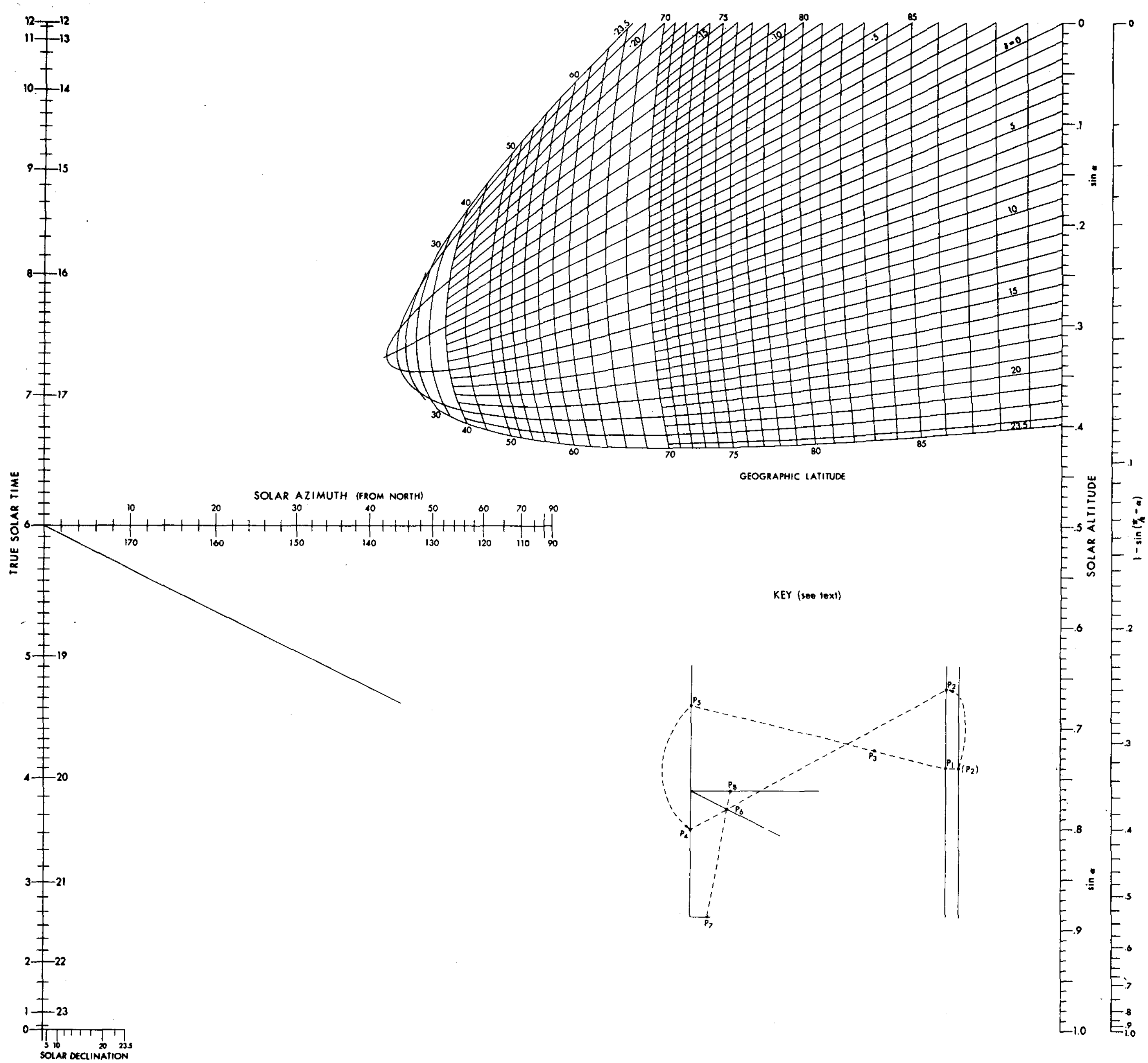

FIGURE 2.—Nomogram for calculation of solar altitude and azimuth for any latitude, solar declination, and time of day.

Once values are assigned to these arbitrary factors, equations (8) may be used to calculate the positions of P_1 ($\sin \alpha$ scale), P_5 ($\cos h$ scale), and P_3 (family of lines of constant ϕ and δ, hereinafter referred to as the $\phi-\delta$ grid). It should be noted that the equations determining X_3 and Y_3 are symmetric in ϕ and δ. It follows that the line in the $\phi-\delta$ grid for $\phi=N°$ is also the line for $\delta=N°$.

For the purposes of the original drafting, the following values were assigned to the arbitrary factors: $K_1=-20$ inches, $C_1=10$ inches, $K_5=10$ inches, $C_5=0$, $X_1=20$ inches. The resulting nomogram is shown in Figure 2.[2]

3. USE OF THE SOLAR ALTITUDE NOMOGRAM

The True Solar Time (also referred to as Local Apparent Time or Sun-Dial Time) determines a point P_5 on the $\cos h$ scale, which is graduated directly in terms of time. The solar declination and geographic latitude determine a point P_3 on the $\phi-\delta$ grid. If P_3 and P_5 are joined by a straight line, which is extended to intersect the $\sin \alpha$ scale

[2] A working size of the nomogram, about 9 in. x 9 in., will be made available in small quantities at no charge to interested readers. Address inquiries to Meteorological Service of Canada, 315 Bloor St. West, Toronto 5, Ontario.

at P_1, the solar altitude (actually sin α) may be read off directly at P_1.

For use at a single station, a nomogram may be constructed with only the latitude line for that station, calibrated re solar declination or directly re time of year. However, for single station work, the graph described by Schutte [2] is easier to construct and more convenient to use.

The solar altitude (sin α) scale may, of course, be labelled directly in terms of any quantity which is a single-valued function of solar altitude; e.g., optical air mass (average refraction assumed), CSCα at 22 km. (for ozone calculations), etc.

4. DISCUSSION OF ERRORS IN THE SOLAR ALTITUDE NOMOGRAM

The errors involved in the use of nomograms may be classified as follows:

(a) Errors in construction (drafting, printing, warping of paper).
(b) Errors in alignment (P_3, P_5).
(c) Errors in reading the answer (P_1).
(d) Curvature of straight-edge used.

The fourth type of error may be reduced by performing the calculation a second time, reversing the straight-edge end for end in doing so, and averaging the two answers.

The errors of types (b) and (c) result from the difficulty in precisely positioning the straight-edge directly over the points P_3 and P_5 and in reading the position of point P_1. In general these errors will depend on the angle at which the straight line P_5P_3 crosses the various scales. If we designate the standard error of positioning the straight line as Δ units of length, perpendicular to the line itself, at P_5 and P_3 and if β is the angle which P_5P_3 makes with the X-axis, then it may be shown that the standard error of P_1 due to alignment error is, in units of sin α,

$$\frac{\Delta \sec \beta}{K_1}\left(1\text{-}2\,\frac{K_1}{K_5}\cos\phi\cos\delta\right). \tag{9}$$

If we consider a further error in the final answer, due to the difficulty of reading off the position of P_1, equal to $\frac{\Delta \sec \beta}{K_1}$, then the final error is

$$\frac{\Delta \sec \beta}{K_1}\sqrt{1+\left(1-2\,\frac{K_1}{K_5}\cos\phi\cos\delta\right)^2}. \tag{10}$$

For the solar altitude nomogram in a convenient working size (about 10 in. x 10 in.), the maximum error occurs when $\phi=\delta=h=0$ and is 0.72Δ. The smallest error occurs when $\phi=90°$, $\beta=0$ and is 0.14Δ.

To provide a practical test of the accuracy obtained in using the nomogram, a set of 25 test computations was designed to sample all areas of the $\phi-\delta$ grid and the time and sin α axes. These were arranged in random order and given to four persons unfamiliar with the nomogram. After a few minutes instruction, each person was asked to go through all computations once, then to repeat the entire set with the straight-edge reversed. The two sets of answers were then compared and any deviations exceeding .007 were reconciled by repeat computations. Finally, the reconciled answers were averaged. The standard error σ, of the individual answer was as follows:

Observer A	(university graduate)	$\sigma=0.013$ (sin α units)
B	(" ")	0.010
C	(high school graduate)	0.013
D	(" " ")	0.012

If the errors in the determinations are classified according to geographic latitude, the following pattern of standard error emerges

$0 \leqq \phi < 20°$	$\sigma=0.027$ (cf. 0.72 Δ)
$20 \leqq \phi < 40$	$\sigma=0.012$
$40 \leqq \phi$	$\sigma=0.005$ (cf. 0.14 Δ)

Thus, neglecting errors in construction, one might infer a value of $\Delta=0.04$ in.

To give some idea of the accuracy obtainable by an experienced observer, one of the authors performed the same set of test computations. The standard error of these computations was 0.003 and a separation of errors according to latitude suggested a value of $\Delta=0.01$ in. Errors of this magnitude are quite acceptable for many meteorological and engineering uses.

5. SOLAR AZIMUTH NOMOGRAM

The solar azimuth angle, a (measured from north), is given by

$$\sin a=\sin h\,\frac{\cos\delta}{\cos\alpha} \tag{11}$$

For solution of this equation, it is convenient to use a proportional chart of the type described by Levens [1]. Such a chart is illustrated in skeleton form in figure 3. From pairs of similar triangles (AP_6P_2 and BP_6P_4, AP_6P_7 and BP_6P_8), it is readily seen that

$$\frac{BP_8}{AP_7}=\frac{BP_4}{AP_2}. \tag{12}$$

Interpreting these distances in terms of the variables (a, h, δ, α) as illustrated, equation (12) is directly equivalent to (11). The proportional chart of figure 3 has been incorporated in the solar altitude nomogram as described below.

The existing solar altitude scale may be used for cos α, noting that

$$1-\cos\alpha=1-\sin(\pi/2-\alpha). \tag{13}$$

An auxiliary scale, calibrated in $1-\sin(\pi/2-\alpha)$ is located immediately adjacent and parallel to the sin α scale. Once sin α is determined, the corresponding value of $1-\sin(\pi/2-\alpha)$ is read off on the adjacent scale and the sin α scale re-entered at this value. Let this point be P_2.

The existing time scale may also be used, since

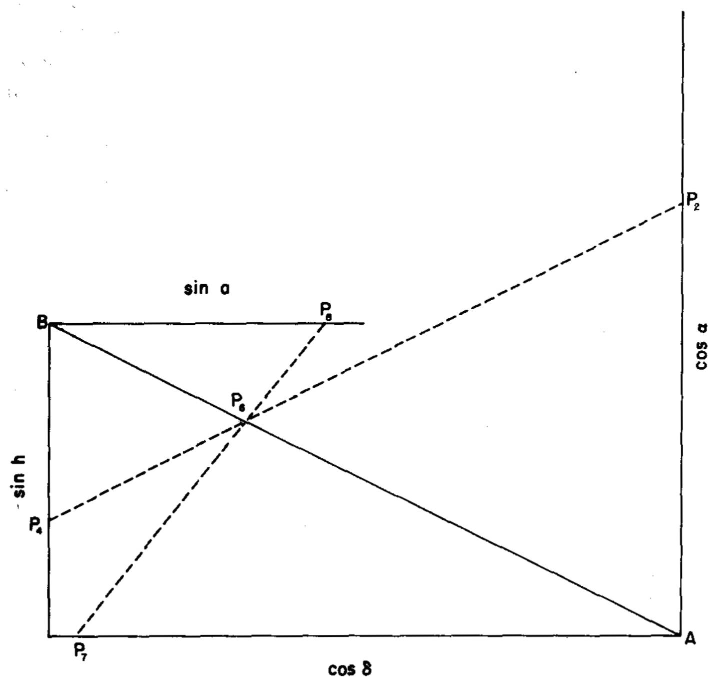

FIGURE 3.—Proportional chart for solution of equation (11).

$$\sin h = \cos(\pi/2 - h) = -\cos(3\pi/2 - h). \tag{14}$$

Only the lower half of the cos h scale is used for the azimuth calculation. To determine the point P_4 at which one enters this scale, one must use either $\pm(h-6 \text{ hr.})$ or $\pm(h-18 \text{ hr.})$, whichever will give a point on the lower half of the time scale.

Points P_2 and P_4 are joined by a straight line which intersects the uncalibrated diagonal at P_6. Then P_6 is joined with P_7 on the solar declination scale at the bottom of the nomogram and the azimuth read off at P_8 on the azimuth scale.

In the Northern Hemisphere, the solar azimuth is measured from north and, during the time of year when $\delta > 0$, confusion may exist as to whether, for example, 70° or 110° should be read off the azimuth scale. When such confusion exists, the azimuth should be determined for a time a little closer to noon. If the new azimuth is closer to the 90° point, then the correct azimuth was 70°. If the new azimuth is slightly farther from the 90° point, then 110° is the correct azimuth.

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